Week 2

Physical Equations

Darcy's Law

Darcy found that the total discharge Q varies in:

- direct proportion to A and to) h (=h_1 - h_2)
- inversely with L

$$Q = KA\frac{h_1 - h_2}{L}$$

where
K = the hydraulic conductivity [L T^{-1}]

The above equation can be rewritten as:

$$\frac{Q}{A} = -K\frac{h_2 - h_1}{L} = -K\frac{h_2 - h_1}{l_2 - l_1}$$

This can be written more generally as:

$$q = -K\frac{dh}{dl}$$

Darcy's equation can be written in terms of head and potential

$$Q = -KA\frac{dh}{dl} = -\frac{KA}{g}\frac{df}{dl}$$

The Darcy's Law is valid only under the laminar flow -- water molecules follow streamlines. The water molecules do not mover along the parallel streamlines in turbulent flow.

Mechanical Energy

Total energy of a unit volume of fluid is the sum of kinetic, gravitational, and fluid pressure energy

$$E_t = \frac{1}{2} rv^2 + rgz + P$$

where D is the density of the fluid, v is the flow velocity, g is the acceleration of the gravity, z is the elevation of center of gravity of the fluid, and P is the pressure.

The above equation can be modified by dividing the Dg on both sides; total energy per unit mass -- **Bernoulli equation**

$$E_{tm} = \frac{1}{2g} v^2 + z + \frac{P}{rg}$$

For steady state flow of a frictionless, incompressible fluid along a smooth line of flow, the total energy per unit mass is constant.

Steady state, incompressible fluid, closed system

1. velocity is very small and ignore kinetic term
2. D not = f(P); it means incompressible
   what is the pressure at point A

   $$f = gz + \frac{[rg(h-z) + p_0] - p_0}{r} = gh$$

   So,

   $$h = z + \frac{P}{rg} = z + j$$

Groundwater Flow Equation

Confined aquifers

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T} \frac{\partial h}{\partial t}$$

The steady state flow has the left hand side equal to zero -- Laplace equation.

Unconfined aquifers

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_y}{T} \frac{\partial h}{\partial t}$$

Boundary Conditions

1. known head (Dirichlet conditions).
2. known flow (Neumann conditions).

3. combinations.

Finite Difference Method for Steady-State Flow (Laplace's Equation)

1. Finite difference grid

Grid coordinates (i, j), Δx, Δy, and Δz.

2. Central approximation

For x direction

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{\dfrac{h_{i+1} - h_{i,j}}{\Delta x} - \dfrac{h_{i,j} - h_{i-1,j}}{\Delta x}}{\Delta x}$$

It can be simplified to

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_{i-1,j} - 2h_{i,j} + h_{i+1,j}}{(\Delta x)^2}$$

For y direction

$$\frac{\partial^2 h}{\partial y^2} \approx \frac{h_{i,j-1} - 2h_{i,j} + h_{i,j+1}}{(\Delta y)^2}$$

The finite difference approximation at the point (i,j) can be described as

$$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j} = 0$$

4. Iterative methods

$$h_{i,j} = \frac{h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}}{4}$$

Jacobi Iteration: the least efficient.

$$h_{i,j}^{m+1} = \frac{h_{i-1,j}^{m} + h_{i+1,j}^{m} + h_{i,j-1}^{m} + h_{i,j+1}^{m}}{4}$$

Gauss-Seidel Iteration: more efficient because of using newly computed head values whenever possible.

$$h_{i,j}^{m+1} = \frac{h_{i-1,j}^{m+1} + h_{i+1,j}^{m+1} + h_{i,j-1}^{m} + h_{i,j+1}^{m}}{4}$$

Successive Over Relaxation (SOR):

The residual c between two successive iterations in Gauss-Seidel method is described as

$$c = h_{i,j}^{m+1} - h_{i,j}^{m}$$

In the SOR method, the new value at point (i,j) can be defined as

$$h_{i,j}^{m+1} = h_{i,j}^{m} + \omega c$$

where ω is the relaxation factor that is larger than 1.

$$h_{i,j}^{m+1} = (1-\omega)h_{i,j}^{m} + \omega\frac{h_{i-1,j}^{m+1} + h_{i+1,j}^{m+1} + h_{i,j-1}^{m} + h_{i,j+1}^{m}}{4}$$

Gauss-Seidel Computer Program: Homework 1.