# Week 3

## Further Application of Laplace's Equations

## Poission's Equation

$$\frac{\partial q_x}{\partial x}\Delta x(b\Delta y)+\frac{\partial q_y}{\partial y}\Delta y(b\Delta x)=R(x,y)\Delta x\Delta y$$

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{R(x,y)}{T}$$

For island recharge in one dimension

$$\frac{\partial^2 h}{\partial x^2}=-\frac{R}{T}$$

The solution for the above equation is

$$h(x)=-\frac{R}{2T}x^2+a_1x+a_2$$

Applying the boundary conditions of h=0 at x=l, and dh/dx=0 at x=0, $a_1$=0 and $a_2=Rl^2/2T$ can be obtained and the above equation can be simplified to

$$h(x)=\frac{R}{2T}(l^2-x^2)$$

Finite Difference Model for Island Recharge

$$\frac{h_{i-1,j}-2h_{i,j}+h_{i+1,j}}{(\Delta x)^2}+\frac{h_{i,j-1}-2h_{i,j}+h_{i,j+1}}{(\Delta y)^2}=-\frac{R}{T}$$

This results in

$$h_{i,j}=\frac{h_{i-1,j}+h_{i+1,j}+h_{i,j-1}+h_{i,j+1}+\Delta x\Delta xR/T}{4}$$

**Finite Difference Method for the Transient Flow**

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R(x,y)}{T}$$

**Explicit finite Difference approximation**

$$\frac{\partial h}{\partial t} \approx \frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{\Delta t}$$ forward time difference approximation

$$\frac{\partial h}{\partial t} \approx \frac{h_{i,j}^{n} - h_{i,j}^{n-1}}{\Delta t}$$ backward time difference approximation

$$\frac{\partial h}{\partial t} \approx \frac{h_{i,j}^{n+1} - h_{i,j}^{n-1}}{2\Delta t}$$ central time difference approximation

The following form can be obtained with the forward finite difference approximation.

$$\frac{h_{i+1,j}^{n} - 2h_{i,j}^{n} + h_{i-1,j}^{n}}{(\Delta x)^2} + \frac{h_{i,j+1}^{n} - 2h_{i,j}^{n} + h_{i,j-1}^{n}}{(\Delta y)^2} = \frac{S}{T}\frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{\Delta t} - \frac{R}{T}$$

This results in

$$h_{i,j}^{n+1} = (1 - \frac{4T\Delta t}{S\Delta x})h_{i,j}^{n} + (\frac{4T\Delta t}{S\Delta x^2})(\frac{h_{i+1,j}^{n} + h_{i-1,j}^{n} + h_{i,j+1}^{n} + h_{i,j-1}^{n}}{4}) + \frac{R_{i,j}^{n}\Delta t}{S}$$

**Aquifer Response to Water Level Change in a Reservoir**

For the one-dimensional case,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t}$$

The explicit finite difference approximation can be obtained

$$\frac{h_{i+1}^{n} - 2h_{i}^{n} + h_{i-1}^{n}}{(\Delta x)^2} = \frac{S}{T}\frac{h_{i}^{n+1} - h_{i}^{n}}{\Delta t}$$

Then

$$h_i^{n+1} = h_i^n + \frac{T\Delta t}{S}\left(\frac{h_{i+1}^n - 2h_i^n + h_{i-1,j}^n}{\Delta x^2}\right)$$

A computer program can be developed based on this algorithm.