Week 4

Two-dimensional finite difference groundwater model (Yu, 1997)

The following PDE governs the transient, two-dimensional ground-water flow in a heterogeneous aquifer:

$$\frac{\partial}{\partial x}(T\frac{\partial h}{\partial x})+\frac{\partial}{\partial y}(T\frac{\partial h}{\partial y}) = S\ \frac{\partial h}{\partial t}+Q_w - Q_n$$

where T is the transmissivity tensor, h is hydraulic head, S_s is specific storage, t is time, Q_w is net groundwater withdrawal rate, and Q_n is a special source or sink rate.

The finite- difference equations can be derived either from a physical method that uses Darcy's law together with the law of mass conservation law or a purely mathematical method that uses finite difference approximations.

Iterative Alternating Direction Implicit Method

According to Yu and Schwartz (1995), Equation 1 can be arranged to be in a finite-difference form for the column calculations

$$a_j h_{i,j-1} + b_j h_{i,j} + c_j h_{i,j+1} = d_j$$

where i and j are the indices of column number and row number at a node, a_j is equal to ($-T_{i, j-1, 1}$), b_j is equal to ($T_{i-1, j, 2}+T_{i, j, 2}+T_{i, j, 1}+T_{i, j-1, 1}+$SÄx^2/Ät$+R_{i, j}$), c_j is equal to ($-T_{i,j,1}$), and d_j is equal to [(SÄx^2/Ät) $h^c_{i, j}- Q_{i, j} + T_{i-1, j, 2}\ h_{i-1, j,k} + T_{i, j, 2}\ h_{i+1, j} + Q_n + R_{i, j} *h_{i, j} + R_{i, j} * h_{i, j}$]. R is recharge factor (R = conductivity/(ÄxÄy)).

An equation comparable to the above equation can be written for every node in the three-dimensional system to yield a set of n equations where n is the number of columns, times the number of rows in each layer. Thus, equations can be assembled to form a set of tridiagonal matrices, which are also diagonally dominant (a_{kk} $ Ó a_{kl} , where k is not equal to l)

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & a_{n1} & b_{n1} & c_{n1} \\ & & & & a_n & b_n \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ h_{n1} \\ h_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ d_{n1} \\ d_n \end{bmatrix}$$

where a_1 and c_n are equal to zero.

Numerical methods for solving the above matrix

Forward solution and backward substitution

An example of 4 by 4 domain for solving the matrix

Forward solution

$$a_j h_{i,j-1} + b_j h_{i,j} + c_j h_{i,j+1} = d_j$$

at the first ith column node, j=1

$$b_1 h_{i,1} + c_1 h_{i,2} = d_1$$

Rearranging the above equation results

$$h_{i,1} = \frac{d_1}{b_1} - \frac{c_1}{b_1} h_{i,2}$$

Make $D_1 = \frac{d_1}{b_1}, C_1 = \frac{c_1}{b_1}$ then

$$h_{i,1} = D_1 - C_1 h_{i,2}$$

Proceed to j=2

$$a_2 h_{i,1} + b_2 h_{i,2} + c_2 h_{i,3} = d_2$$

Solve $h_{i,2}$ from the above equation

$$b_2 h_{i,2} = d_2 - a_2 h_{i,1} - c_2 h_{i,3}$$

Replacing $h_{i,1}$ from the previous equation results in

$$b_2 h_{i,2} = d_2 - a_2 (D_1 - C_1 h_{i,2}) - c_2 h_{i,3}$$

Rearranging the equation, we obtain

$$h_{i,2} = \frac{d_2 - a_2 D_1}{b_2 - a_2 C_1} - \frac{c_2}{b_2 - a_2 C_1} h_{i,3}$$

Same as the above equation, make $D_2 = \frac{d_2 - a_2 D_1}{b_2 - a_2 C_1}, C_2 = \frac{c_2}{b_2 - a_2 C_1}$

We have

$$h_{i,2} = D_2 - C_2 h_{i,3}$$

Using the same way, we can obtain

$$h_{i,3} = D_3 - C_3 h_{i,4}$$

and

$$h_{i,4} = D_4 - C_4 h_{i,5}$$

Because there is no $h_{i,5}$, it equals to zero. So

$$h_{i,4} = D_4$$

Backward Substitution

Because the $h_{i,4}$ is known now, we can substitute into equation for solving the $h_{i,3}$, then $h_{i,2}$, and $h_{i,1}$.

Computer Program

Segment 1. Forward solution

```
Do j=1, nr
    ......
    W=bb-aa*C(j-1)
```

```
        C(j)=cc/W
        D(j)=(dd-aa*D(j-1))/W
        ……
    Enddo
```

Segment 2. Backward substitution

```
    Do j=nr, 1
        ……
        h(i,j)=(D(j)-C(j)h(i,j-1))
        ……
    Enddo
```